MATH 1ZB3

Gulkaran Singh

Winter 2023

Contents

1 Type 1 Improper Ingegrals

These are the integrals that approach infinity. We say that if $f(x)$ is continuous on $[a, \infty)$ then,

$$\int_a^{\infty} f(x)\ dx = \lim_{b \to \infty} \int_a^{b} f(x)\ dx \tag{1}$$

and vise versa for $-\infty$

$$\int_{-\infty}^{b} f(x)\ dx = \lim_{a \to -\infty} \int_a^{b} f(x)\ dx \tag{2}$$

If the limit exists, we call this integral **convergent**. Otherwise, if the limit D.N.E or approaches infinity then it is **divergent**.

1.1 Extended Type 1

Say we have an integral from $-\infty$ to ∞, then we can break the integral up into two integrals as such,

$$\int_{-\infty}^{\infty} f(x)\ dx = \int_a^{\infty} f(x)\ dx + \int_{-\infty}^{a} f(x)\ dx \tag{3}$$

Here both limits need to exist for it to be convergent. If **either** limit D.N.E, then it diverges.

Solving Questions: typically you just take the limit, integrate, and substitute infinity.

2 Type 1 p-integrals

These are the integrals in the form

$$\int_1^{\infty} \frac{1}{x^p}\ dx \tag{4}$$

They **converge** for $p > 1$ and diverge to infinity otherwise ($p \leq 1$).

Remark: only the behaviour at $\pm\infty$ determines convergence. So if the integral doesn't have a lower limit of 1, break it up into 2 integrals and notice how the integral containing infinity determines convergence.

3 Type 2 Improper Integrals

These are the integrals that approach from an asymptote. If $f(x)$ is continuous on $(a, b]$ – a would be the asymptote – then,

$$\int_a^b f(x)\ dx = \lim_{c \to a^+} \int_c^b f(x)\ dx \tag{5}$$

and vise versa if b is the asymptote

$$\int_a^b f(x)\ dx = \lim_{c \to b^-} \int_a^c f(x)\ dx \tag{6}$$

Similarly, if the limit exists the integral converges, otherwise it diverges.

4 Type 2 p-integrals

These are the integrals in the form

$$\int_0^1 \frac{1}{x^p}\ dx \tag{7}$$

They **converge** for $p < 1$ and diverge to infinity otherwise ($p \geq 1$).

Notice: the convergence condition switched!

5 Improper Integrals & Comparisons

We can use the **comparison** test to determine convergence or divergence. This method lets us compare the convergence/divergence of an easier but similar integral to our original one.

$$\text{If } 0 < f(x) \leq g(x), \text{ and } \int_a^\infty g(x)\ dx \textbf{ converges} \text{ then } \int_a^\infty f(x)\ dx \textbf{ converges}$$

$$\text{If } f(x) \geq g(x) > 0, \text{ and } \int_a^\infty g(x)\ dx \textbf{ diverges} \text{ then } \int_a^\infty f(x)\ dx \textbf{ diverges}$$

Tip: Greater than divergent, diverges. Less than convergent, converges.

6 Sequences

A **sequence** is an infinite ordered list of values, denoted by $\{a_n\}_{n=1}^{\infty}$. There are two ways to define sequences.

(1) **explicit** sequences – expressed as $a_n = f(n)$

$$\text{e.g.} \quad a_n = n^3, \quad b_n = \frac{1}{\ln n + 1}$$

(2) **recursive** sequences – expressed by a *recurrence relation* in terms of previous terms and initial conditions.

$$\text{e.g.} \quad a_{n+1} = 2a_n,\ a_1 = 2, \qquad b_{n+1} = \sqrt{3 + b_n},\ b_1 = 1$$

7 Convergence of a Sequence

We can determine convergence/divergence of a sequence by taking its limit

$$\lim_{n\to\infty} a_n = L \in \mathbb{R} \tag{1}$$

If the limit exists at L then it is **convergent** and otherwise divergent.

7.1 Alternating Sequences

Similar to the concept in **11.5**, an alternating sequence a_n will converge <u>**iff**</u>

$$\lim_{n\to\infty} |a_n| = 0 \tag{2}$$

Questions usually have alternating signs in the form $(-1)^n$ or similar. Taking the absolute value of these just means removing the negative signs and keeping all else.

8 Sequences & Recurrence

For recursively defined sequences, we have to assume the limit exists. If it does, then its a fixed point of the recurrence relation.

$$\lim_{n\to\infty} a_{n+1} = f(L) = L \tag{8}$$

Remark: we say $a_{n+1} = f(a_n)$ and taking the limit of a_n gives us L which is where we get $f(L)$ from. The other L is what we get from $\lim_{n\to\infty} a_{n+1}$

Video Explanation: explaining the formula + how to choose L

9 Convergence from Constraint

We can say a sequence converges if it is **bounded**

$$a < a_n < b \tag{9}$$

and **monotomic** – traveling in one direction

$$\begin{array}{ccc} a_{n+1} \geq a_n & \text{or} & a_{n+1} \leq a_n \\ \text{(increasing)} & & \text{(decreasing)} \end{array} \tag{10}$$

10 Convergence & Induction

Proof by induction is a proof method that will help us determine if a statement is true for every natural number. Its two parts are

(1) The Base Case – our statement holds for an initial case

(2) The "Induction Step" – show our statement holds for an arbitrary (random) kth case, as well as the next $k+1$ case.

Questions: they will typically ask to show that a recursive sequence is bounded or monotomically increasing/decreasing.

- show **bounded** – pick $a \leq a_k \leq b$ and do operations to make a_k into a_{n+1}
- show **monotomic** – show $a_{k+1} \geq a_k$ or $a_{k+1} \leq a_k$ depending on if the question asks to prove increasing or decreasing. Then do operations on the inequality to show $a_{k+2} \geq a_{k+1}$ or $a_{k+2} \leq a_{k+1}$

11 Series

A series is an **infinite** sum and works similar to improper integrals

$$\sum_{n=1}^{\infty} a_n = \lim_{m \to \infty} \sum_{n=1}^{m} a_n \tag{10}$$

and similarly, if the limit exists it is convergent and divergent otherwise.

If $S_m = \sum_{n=1}^{m}$ is the m**th partial sum**, then

$$S_\infty = \lim_{m \to \infty} S_m = S \tag{11}$$

For example, S_3 would be the 3rd partial sum and would equate to $a_1 + a_2 + a_3$

11.1 Geometric Series Formula

If we know the formula for a geometric sum

$$\sum_{i=1}^{n} ar^{i-1} = \frac{a(1-r^n)}{1-r}$$

then we can derive the formula of a geometric series by applying a limit to ∞

$$\lim_{n\to\infty} \sum_{i=1}^{n} ar^{i-1}$$

$$= \lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$$

$$= \begin{cases} \frac{a}{1-r} & |r| < 1 \\ \infty & |r| > 1 \text{ and } a > 0 \\ -\infty & |r| > 1 \text{ and } a < 0 \end{cases}$$

To find an expression for a specific term, take the difference of the partial sum of that term and the one before

$$a_n = S_n - S_{n-1} \tag{12}$$

11.2 Divergence Test

To check for divergence of a series, we do the **divergence test**

$$\text{If } \lim_{n\to\infty} a_n \neq 0, \text{ then } \sum_{n=1}^{\infty} a_n \textbf{ must} \text{ diverge} \tag{13}$$

12 Improper Integrals & Series

Another way to check convergence/divergence of a series is by noticing the similarity between series and improper integrals.

12.1 The Integral-Series Comparison Test

If a series $\sum_{n=1}^{\infty} a_n$ where $a_n = f(n)$ and is **positive**, **continuous**, and **decreasing** then

$$\sum_{n=1}^{\infty} a_n \text{ converges if } \int_1^{\infty} f(x)\, dx \text{ converges} \tag{14}$$

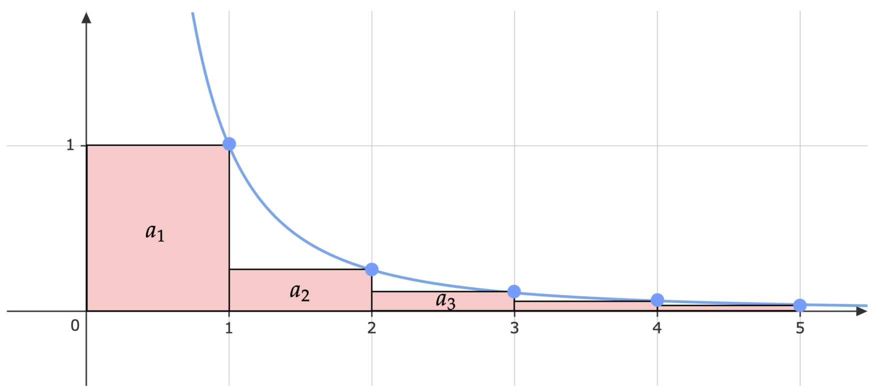
$$\sum_{n=1}^{\infty} a_n \text{ diverges if } \int_1^{\infty} f(x)\, dx \text{ diverges} \tag{15}$$

12.2 Similarly, p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{converges for } p > 1 \\ \text{diverges for } p \leq 1 \end{cases} \tag{16}$$

13 Integral Approximations

Consider a convergent series, we can say the remainder $R_m = S - S_m$



This R_m is less than the integral from m to ∞ as the area blocks are a_n high but 1 wide (not infinitely thin). We call this the **upper bound on the remainder**

13.1 Integral Error Estimate

For a convergent series

$$R_m = S - S_m \leq \int_m^{\infty} f(x)\ dx \tag{17}$$

Lecture approximation questions are as simple as making $\int_m^{\infty} \leq \epsilon$ and to find what m should be. Notice, we always round up!

14 Comparison Tests

Similar to the comparison tests for improper integrals, we have the same test for series

14.1 Series-Series Comparison Test

Given two infinite series where $0 \leq a_n \leq b_n$

$$\sum_{n=1}^{\infty} a_n \text{ converges if } \sum_{n=1}^{\infty} b_n \text{ converges} \tag{18}$$

$$\sum_{n=1}^{\infty} a_n \text{ diverges if } \sum_{n=1}^{\infty} b_n \text{ diverges} \tag{19}$$

Recall: the inequality direction matters! Less than convergent converges and greater than divergent diverges.

14.2 Series Limit Comparison Test

Consider taking the limit of

$$\lim_{n \to \infty} \frac{a_n}{b_n} = L \text{ where } 0 < L < \infty \tag{20}$$

then the same result as above applies here too. Both series will converge or diverge but not to the same value.

15 Alternating Tests

An alternating series is in the form

$$\sum_{n=1}^{\infty} (-1)^n b_n \quad \text{ or } \quad \sum_{n=1}^{\infty} (-1)^{n+1} b_n$$

Note: there are many ways the $(-1)^n$ can be represented. Here are some very common patterns

- $(-1)^n = (-1)^{n \,\pm\, \text{even}} = \cos(n\pi)$
- $(-1)^{n+1} = -(-1)^n = (-1)^{n-1} = (-1)^{n \,\pm\, \text{odd}}$

15.1 Alternating Series Test (AST)

An alternating series will **converge** if

$$b_n > 0,\ b_{n+1} \leq b_n \text{ (monotomically decreasing), and } \lim_{n \to \infty} b_n = 0 \tag{21}$$

Tip: to start doing questions with AST, extract the signed component $(-1)^n$ and consider the rest as b_n.

16 Alternating Series Approximations

For a convergent alternating series

$$|R_m| = |S - S_m| \leq b_{m+1} \tag{22}$$

Tip: to find the **upper bound**, you simply just find b_{m+1}

Lecture approximation questions are as simple as making $b_{m+1} \leq \epsilon$ and plugging in the expression for b_{m+1} to find what m should be.

17 11.5 Absolute Convergence

This is a sub-type of convergence where we take the absolute value of the terms.

$$\text{If } \sum_{n=1}^{\infty} |a_n| \text{ converges, then } \sum_{n=1}^{\infty} a_n \text{ MUST converge} \tag{23}$$

We say the series **converges absolutely**. If $\sum_{n=1}^{\infty} |a_n|$ diverges, but $\sum_{n=1}^{\infty} a_n$ converges, then the series is said to be **conditionally convergent.**

18 Ratio & Root Tests

Another method to determine the convergence/divergence of an arbitrary series.

18.1 The Ratio Test

Given a series $\sum_{n=1}^{\infty} a_n$, compute

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L \quad \begin{cases} \text{if } L < 1 & \text{then converges} \\ \text{if } L > 1 & \text{then diverges} \\ \text{if } L = 1 & \text{then inconclusive} \end{cases} \tag{24}$$

Remark: this test fails when a_n grows at a polynomial or slower rate. It works best for geometric or faster. Really good for factorials!

18.2 The Root Test

Similar to the ratio test, the root test looks at the nth root of the series.

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L \quad \begin{cases} \text{if } L < 1 & \text{then converges} \\ \text{if } L > 1 & \text{then diverges} \\ \text{if } L = 1 & \text{then inconclusive} \end{cases} \tag{25}$$

Tip: use when your a_n is risen to the power of n, that way the nth root can be written as $\frac{1}{n}$ and cancel the initial power.

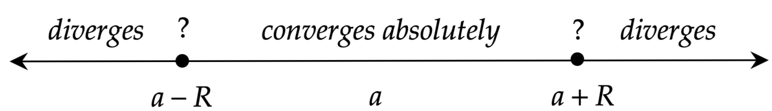
19 Power Series

These are series with powers of x or $x - a$. The a is called the **centre** of the series, and the domain of $x \in \mathbb{R}$ where the series converges is called the **interval of convergence**.

$$\sum_{n=0}^{\infty} c_n x^n \quad \text{or} \quad \sum_{n=0}^{\infty} c_n (x-a)^n \tag{26}$$

Note: the first is called a '0-centred power series' whilst the second is called 'a-centred power series'.

We typically start by doing a ratio test on the power series. The result (previously L) is called the **radius of convergence** and is denoted as R.



At these ? points, we don't know if the power series converges or diverges. We plug R in for x in our power series and do a limit comparison test if the series is positive, or ATS if its alternating series.

19.1 Edge Cases

What happens when our ratio test limit approaches ∞ or 0?

- if the limit $\to \infty$, since $\infty > 1$ it will diverge everywhere except at $x = 0$, so our $R = 0$
- if the limit $\to 0$, since $0 < 1$, our converge everywhere so our $R = \infty$

20 MacLaurin & Taylor Polynomials

These power series of polynomials help us approximate any type of function. For 0-centered functions we have the MacLaurin Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

and similarly for a power series about $x = a$,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

The most common type of question revolves around finding the MacLaurin or Taylor Series for a given $f(x)$. We start by trying to find a pattern with its $n = 0, 1, 2, \cdots$ derivative

20.0.1 Example

Find the Taylor series for $f(x) = \frac{1}{\sqrt{x}}$ centered at $a = 4$

n	$f^{(n)}(x)$	$f^{(n)}(x)$
0	$x^{-1/2}$	$\frac{1}{2}$
1	$x^{-3/2}\left(-\frac{1}{2}\right)$	$\left(\frac{1}{8}\right)\left(-\frac{1}{2}\right) = \frac{1}{2^4}$
2	$x^{-5/2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)$	$\left(\frac{1}{32}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) = \frac{1}{2^7}$
3	$x^{-7/2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)$	$\left(\frac{1}{128}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right) = \frac{1}{2^{10}}$

Now we can start filling in the formula from the patterns we notice

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(4)}{n!}(x-4)^n \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!\ 2^{3n+1}}(x-4)^n \end{aligned}$$

Notice that we can even increment the sum to start at $n = 1$ by adding the first term seperately. From there we can even factor out that first term to get an alternate form.

$$\begin{aligned} &= \frac{1}{2} + \sum_{n=1}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!\ 2^{3n+1}}(x-4)^n \\ &= \frac{1}{2}\left[1 + \sum_{n=1}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!\ 2^{3n}}(x-4)^n\right] \end{aligned}$$

20.1 Famous Taylor Series

Here are the most common and popular Taylor Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

21 Taylor Polynomials & Remainders

We can find the *partial* sums for a Taylor Series and we call this a Taylor Polynomial, denoted by $T_m(x)$.

$$\begin{aligned} T_m(x) &= \sum_{n=0}^{m} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(m)}(a)}{m!}(x-a)^m \end{aligned}$$

These questions are done the same way but this time you can simplify.

21.1 Taylor Remainder Theorem

To find the upper bound or maximum error we use the following formula

$$|f(x) - T_m(x)| = R_m(x) \leq \frac{M|x-a|^{m+1}}{(m+1)!}, \quad M \geq \sup |f^{(m+1)}(x)|$$

22 Binomial Series

Starting off with binomial coefficients, we know their formula is

$$(1+x)^m = \sum_{n=0}^{m} \binom{m}{n} 1^{m-n} x^n = 1 + \sum_{n=1}^{m} \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!} x^n$$

22.0.1 Example

Use the binomial series to find the MacLaurin series for

$$f(x) = \frac{1}{(2+x^2)^2}$$

First, we want to rewrite $f(x)$ so it looks like $(1+x)^m$ using u-substitution

$$\begin{aligned} f(x) &= (2+x^2)^{-2} \\ &= (2(1+\frac{x^2}{2}))^{-2} \\ &= \frac{1}{4}(1+\frac{x^2}{2})^{-2} \\ \text{let } u &= \frac{x^2}{2} \text{ and } m = -2 \\ &= \frac{1}{4}(1+u)^m \end{aligned}$$

Now we can use the formula and sub back m

$$\begin{aligned}\frac{1}{4}(1+u)^m &= \frac{1}{4}\sum_{n=0}^{m}\frac{m(m-1)(m-2)\cdots(m-n+1)}{n!}u^n\\ &= \frac{1}{4}\sum_{n=0}^{m}\frac{(=2)(-3)(-4)\cdots((-2)-n+1)}{n!}u^n\end{aligned}$$

Notice the numerator simplifies to $(-1)^n(n+1)!$. We can also sub back in u.

$$\begin{aligned}&= \frac{1}{2^2}\sum_{n=0}^{m}\frac{(-1)^n(n+1)!}{n!}\left(\frac{x^2}{2}\right)^n\\ &= \sum_{n=0}^{m}\frac{(-1)^n(n+1)}{2^{n+2}}x^{2n}\end{aligned}$$

23 Surface Area of Revolution

We combine the volume and arc-length formulas to get the SA formula about the x-axis

$$SA = \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\ dx$$

and about the y-axis

$$SA = \int_a^b 2\pi {\color{red}x}\sqrt{1+[f'(x)]^2}\ dx$$

The key idea to complete these questions is knowing how to complete the square so you can rewrite $1+[f'(x)]^2$ as $(a+b)^2$ so it cancels with the square root.

23.0.1 Example

Find the area of the surface obtained by rotating the curve $y = \frac{x^3}{6}+\frac{1}{2x}, 1 \le x \le 2$ about the y-axis.

First we start with finding $1+[f'(x)]^2$ and try to rewrite it in terms of a square

$$\begin{aligned}f'(x) &= \frac{1}{2}x^2 - \frac{1}{2x^2}\\ [f'(x)]^2 &= \frac{1}{4}x^4 - \frac{1}{2} + \frac{1}{4x^4}\\ 1+[f'(x)]^2 &= \frac{1}{4}x^4 - \frac{1}{2} + \frac{1}{4x^4} + 1\\ &= \frac{1}{4}x^4 + \frac{1}{2} + \frac{1}{4x^4}\\ &= \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2\end{aligned}$$

From here you can very easily plug that into the integral and solve.

24 Ordinary Differential Equations

These are differential equations with only **one** independant variable. The *solution* to O.D.E's is finding the y that solves/satisfies both sides of the O.D.E. The **order** of an O.D.E is the highest derivative. For example, the solution to $y' = \cos(x)$ would be $y = \sin(x)$.

24.0.1 Example

For what value of m is $y = x^m$ a solution to the differential equation $8x^2y'' + 6xy' - 3y = 0$?

Start with finding each part of the equation with the information we know.

$$\begin{aligned}
-3y &= -3(x^m) \\
6xy' &= 6x(mx^{m-1}) \\
&= 6m(x^m) \\
8x^2y'' &= 8x^2(m(m-1)x^{m-2}) \\
&= (8m^2 - 8m)(x^m)
\end{aligned}$$

Now we can substitute back into the equation and factor out the common x^m

$$\begin{aligned}
0 &= (8m^2 - 8m)(x^m) + 6m(x^m) - 3(x^m) \\
&= x^m(8m^2 - 8m + 6m - 3) \\
&= x^m(4m-3)(2m+1)
\end{aligned}$$

Therefore, the values of m that satisfy the equation are $\frac{3}{4}$ and $-\frac{1}{2}$.

25 Seperable O.D.Es

These are first order O.D.Es in the form

$$y' = f(x)g(y) \iff \int \frac{1}{g(y)}\ dy = \int f(x)\ dx$$

We are able to do this because we are taking the integral w.r.t the independant variable

$$\begin{aligned}
y' &= \frac{x}{y} \\
y'y &= x \\
\int \frac{dy}{\cancel{dx}} y\ \cancel{dx} &= \int x\ dx
\end{aligned}$$

25.1 Orthagonal Trajectories

These questions all have the same 5 step process to solve.

(1) Implicit Differentiation to find ${y'}_{old}$

(2) Find k from original equation

(3) Plug k back in to complete finding ${y'}_{old}$

(4) Use the fact that ${y'}_{new} = -1/{y'}_{old}$

(5) Solve the O.D.E

25.1.1 Example

Find the orthoganal trajectories of the family of curves

$$y = \frac{1}{(x+k)^3}$$

To find the orthoganal trajectories of this family of curves, we first start by implicit differentiation

$$y' = -\frac{3}{(x+k)^4}$$

From here, we want to use our original equation to find an expression for k.

$$\begin{aligned} y &= \frac{1}{(x+k)^3} \\ k &= \frac{1}{y^{\frac{1}{3}}} - x \end{aligned}$$

Now we can sub this k into our y' equation we have,

$$\begin{aligned} y' &= -\frac{3}{(x+(y^{-\frac{1}{3}}-x))^4} \\ &= -\frac{3}{(y^{-\frac{4}{3}})} \\ y' &= -3y^{\frac{4}{3}} \end{aligned}$$

This is what we will label y'_{old}, to find the orthoganal trajectories, we use the following formula,

$$\begin{aligned} y'_{\text{new}} &= -\frac{1}{y'_{\text{old}}} \\ &= -\frac{1}{-3y^{\frac{4}{3}}} \\ y' &= \frac{1}{3y^{\frac{4}{3}}} \end{aligned}$$

As you may be able to notice, this is set up as a seperable first order O.D.E, so we can find y with the following,

$$\begin{aligned}
y'y^{\frac{4}{3}} &= \frac{1}{3} \\
\int \frac{dy}{dx}y^{\frac{4}{3}}\ dx &= \int \frac{1}{3}\ dx \\
\int y^{\frac{4}{3}}\ dy &= \int \frac{1}{3}\ dx \\
\frac{3}{7}y^{\frac{7}{3}} &= \frac{1}{3}x + C \\
y^{\frac{7}{3}} &= \frac{7}{9}x + C \\
y &= \left(\frac{7}{9}x + C\right)^{\frac{3}{7}}
\end{aligned}$$

26 Exponential Growth

We can model expoential growth as a seperable O.D.E with the form

$$y = y_0 e^{kt}$$

where y_0 is the initial value, k is the rate of growth, and t is the time.

One special type of k value is known as the half-life,

$$\begin{aligned}
\lambda &= -\frac{\ln(2)}{k} \\
k &= -\frac{\ln(2)}{\lambda} \\
y &= y_0 e^{-\frac{\ln(2)}{\lambda}t}
\end{aligned}$$

We also can model temperature questions where we let $y = T - T_{env}$

$$\begin{aligned}
y &= y_0 e^{kt} \\
T - T_{env} &= (T_0 - T_{env})e^{kt}
\end{aligned}$$

27 Linear O.D.Es

These are first order O.D.E's in the form $y' + P(x)y = Q(x)$. The whole idea with these is that we can rewrite them as a product rule and solve using our seperable O.D.E methods. We typically need to multiply through by $I(x)$ so it gets into that form of the product rule,

$$I(x) = e^{\int P(x)}$$

27.0.1 Example

Let $y(x)$ be the solution to the following initial value problem. Find $y(e)$.

$$x^4y' + 5x^3y = \frac{\ln(x)}{x}, \quad x > 0, \quad y(1) = 2$$

Notice how $x^4y' + 5x^3y = \frac{\ln(x)}{x}$ is almost a first order linear ODE. If we divide by x^4 to isolate y', we can get it in the correct form.

$$y' + \frac{5}{x}y = \frac{\ln(x)}{x^5}$$

Also notice that we can label $P(x) = \frac{5}{x}$ and $Q(x) = \frac{\ln(x)}{x^5}$. Now we can find $I(x)$ with the following,

$$\begin{aligned} I(x) &= e^{\int P(x)dx} \\ &= e^{\int \frac{5}{x}dx} \\ &= e^{5\ln(x)} \\ &= e^{\ln(x^5)} \\ I(x) &= x^5 \end{aligned}$$

Now we can multiply $I(x)$ through the first order linear ODE and solve for y.

$$\begin{aligned} x^5(y' + \frac{5}{x}y) &= x^5\left(\frac{\ln(x)}{x^5}\right) \\ x^5y' + 5x^4y &= \ln(x) \\ \frac{d}{dx}(x^5y) &= \ln(x) \\ x^5y &= \int \ln(x)dx \\ &= x\ln(x) - \int dx \\ &= x(\ln(x) - 1) + C \\ &= x\ln(x) - x + C \\ y &= \frac{x(\ln(x) - 1) + C}{x^5} \end{aligned}$$

Now we can use the initial value of $y(1) = 2$ to solve for C.

$$\begin{aligned} 2 &= \frac{(1)(\ln(1) - 1) + C}{(1)^5} \\ 2 &= -1 + C \\ 3 &= C \end{aligned}$$

Finally, we can find $y(e)$.

$$y(e) = \frac{e(\ln(e) - 1) + 3}{(e)^5}$$
$$\therefore y(e) = \frac{3}{e^5}$$

28 Parametric Equations

This type of equation has x and y as functions $f(t)$ and $g(t)$ respectively which lets each t be represented by the point

$$(x, y) = (f(t), g(t))$$

which traces a parametric curve. t is determined by the output of these two functions in a vector like fashion, so parametric curves have a direction and orientation. To draw these out, you simply start with increasing t values and see what the corresponding (x, y) point would be given from the $f(t), g(t)$ functions.

28.1 Parametric Calculus

To find the derivative of a parametric equation, we have the following

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$$

The integral of a parametric equation is given by the following

$$\int_a^b f(x)\ dx = \int_{t_0}^{t_1} y(t)x'(t)\ dt$$

The arc length of a parametric equation can be found with

$$L = \int_{t_0}^{t_1} \sqrt{(x'(t))^2 + (y'(t))^2}\ dt$$

To find the surface area around the x-axis we can use

$$SA = \int_{t_0}^{t_1} 2\pi y(t) \sqrt{(x'(t))^2 + (y'(t))^2}\ dt$$

and for the y-axis

$$SA = \int_{t_0}^{t_1} 2\pi x(t) \sqrt{(x'(t))^2 + (y'(t))^2}\ dt$$

28.1.1 Example

The parametric curve $x = 2t^4 + 3t^2 + 1$ and $y = 4t^3 - 4t$ have two tangent lines at the point (6,0). Find their slopes.

To solve for the slope, we need to evaluate the derivative of the parametric curve at the t value that results in the point (6,0). Start by finding the derivative of the curve,

$$\begin{aligned}\frac{dy}{dx} &= \frac{y'(t)}{x'(t)} \\ &= \frac{12t^2 - 4}{8t^3 + 6t}\end{aligned}$$

Now we can find the corresponding t value that gets us our point (6, 0). Start with finding what t makes $y(t) = 0$ and use that to match it to the t that will make $x(t) = 6$

$$\begin{aligned}y(t) &= 0 \\ 0 &= 4t^3 - 4t \\ &= 4t(t^2 - 1) \\ \therefore t &= 0 \text{ and } t = \pm 1\end{aligned}$$

Now let's evaluate $x(t)$ at $t = 0$ and $t = \pm 1$ to find which t makes $x(t) = 6$.

$$\begin{aligned}x(0) &= 2(0)^4 + 3(0)^2 + 1 = 1 \\ x(1) &= 2(1)^4 + 3(1)^2 + 1 = 6 \\ x(-1) &= 2(-1)^4 + 3(-1)^2 + 1 = 6\end{aligned}$$

From this we can at $t = \pm 1$ results in the point (6,0) on our parametric curve. Now let's evaluate our derivative at $t = \pm 1$.

$$\begin{aligned}\left.\frac{dy}{dx}\right|_{t=1} &= \frac{4}{7} \\ \left.\frac{dy}{dx}\right|_{t=-1} &= -\frac{4}{7}\end{aligned}$$

29 Polar Coordinates

Polar coordinates are in the form (r, θ) where r is the magnitude of the vector with angle θ. Here are the key identities

$$\begin{aligned}r^2 &= x^2 + y^2 \\ x &= r\cos(\theta) \\ y &= r\sin(\theta) \\ \tan(\theta) &= y/x\end{aligned}$$

To change from r to $-r$ or vise versa, add π to the angle.

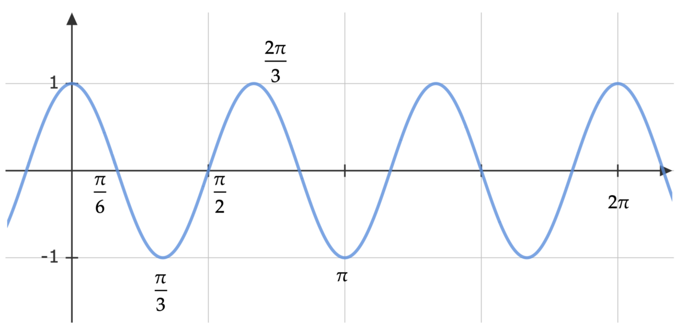
29.1 Polar Graphing

To graph polar functions we use a table of values. We can use the graph of $y = f(x)$ instead of $r = f(\theta)$ to determine the behaviour of the graph.

29.1.1 Example

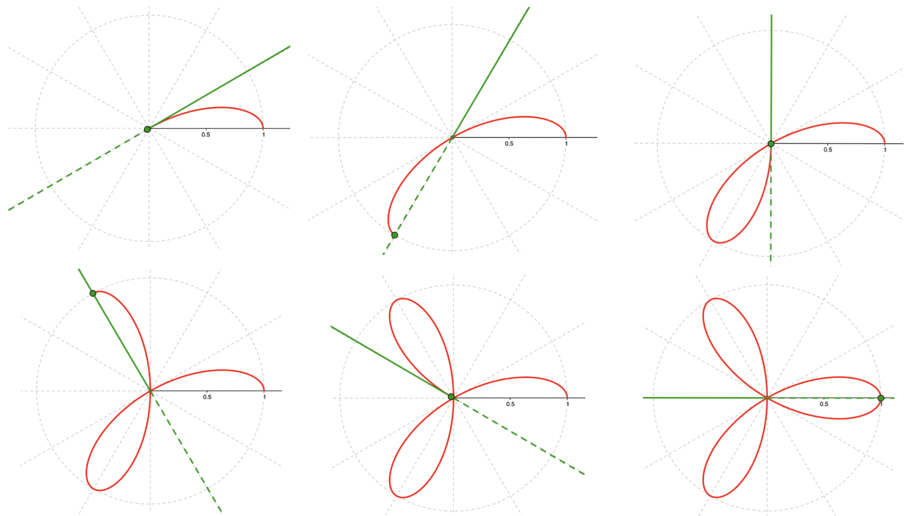
Graph $r = \cos(3\theta)$ by using the graph $y = \cos(3x)$.

We start with the graph of $y = \cos(3x)$, from here we can see that our starting **polar** point is (1, 0) as $\theta = x = 0$ and $r = y = 1$. We can make a table of values from here.

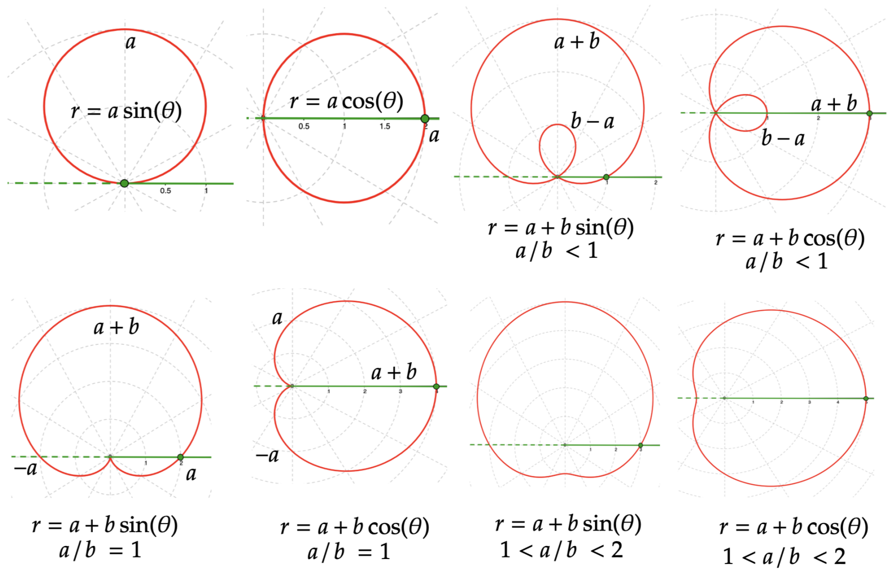


θ	r
0	1
$\pi/6$	0
$\pi/3$	-1
$\pi/2$	0
$2\pi/3$	1

From this we can start from (1,0) and go to magnitude 0 by gradually hitting the $\pi/6$ angle. Here's what the full process looks like



There are also some common forms to look out for.



29.2 Polar Derivatives

It's as simple as plug and chug into this formula

$$\begin{aligned}\frac{dy}{dx} &= \frac{y'}{x'} \\ &= \frac{(r\sin(\theta))'}{(r\cos(\theta))'} \\ &= \frac{r'\sin(\theta) + r\cos(\theta)}{r'\cos(\theta) - r\sin(\theta)}\end{aligned}$$

30 Multivariable Functions

Let's take a look at functions with more than one independant variable. For example, $f(x, y)$

- has 2 independant variables (x, y)
- its graph would be plotted in Cartesian $\mathbb{R}^3$
- the domain is equivalent to a region on the xy plane a set of (x, y) values
- we can write the rectangular domain as $(x, y) \in [a, b] \times [c, d]$ where $[a, b]$ is the x-interval and $[c, d]$ is the y interval

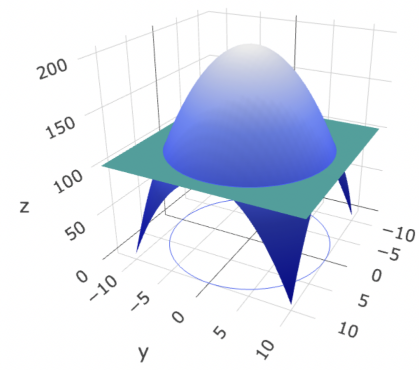
31 Level Sets

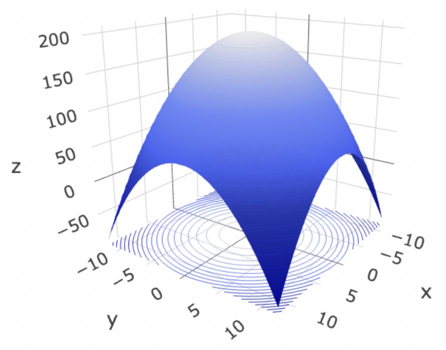
Level sets help up visualize the 3d aspect of our graphs. Imagine we take a slice through our graph at $z = 0, 1, 2$, we bring that shape down to our domain (to the xy plane) and we have our level sets. Be warned, level sets

- must be in the domain of $f(x, y)$
- for different k values cannot cross for your graph to be a function

Finding level sets is easy. Replace $f(x, y)$ with k and try to rewrite the function interms of y.

31.0.1 Visualization



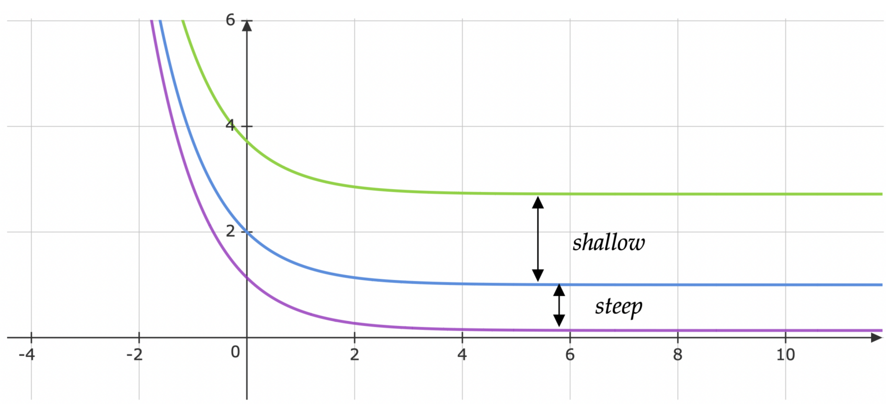


31.0.2 Example

Find level sets for $f(x, y) = \ln(y - e^{-x})$

$$k = \ln(y - e^{-x})$$
$$y = e^{-x} + e^k$$

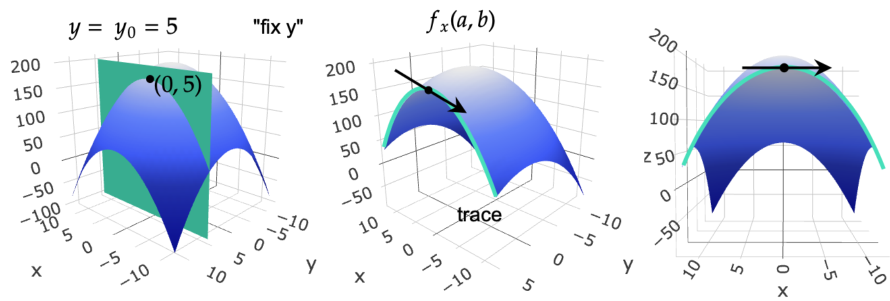
Notice, e^k is just a constant for different k values, so if we chose $k = 0, 1, 2, 3$, here's what it looks like graphed



32 Partial Derivatives

To calculate the derivative of $f(x, y)$, we set one of the variables **constant** so we can calculate the derivative of its trace.

32.0.1 Visualization



Note we can do this same process to x as well and 'fix' x. Therefore, we have

$$f_x(x, y) = \frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \frac{\partial z}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

To calculate the partial derivative, derive normally with respect to the chosen variable and keep the other variables constant!

32.0.2 Example

Find the first partial derivative w.r.t x of $f(x, y, z) = \dfrac{z \tan^{-1}(xy)}{y}$

$$f_x(x, y, z) = \frac{\partial}{\partial x}\left(\frac{z \tan^{-1}(xy)}{y}\right)$$

We can pull out the z/y as constant,

$$= \frac{z}{y}\left(\frac{\partial}{\partial x}(\tan^{-1}(xy))\right)$$

Remember, y acts like any other constant, so this is inheriently chain rule

$$\begin{aligned}
&= \frac{z}{y}\left(\frac{1}{1+(xy)^2}\right)\left(\frac{\partial}{\partial x}(xy)\right) \\
&= \frac{z}{y}\left(\frac{y}{1+(xy)^2}\right) \\
\therefore f_x(x, y, z) &= \frac{z}{1+x^2y^2}
\end{aligned}$$

33 Higher Derivatives

We can take second partial derivatives in x and y to provide information on concavity.

$$f_{xx}(x,y) = \frac{\partial^2 z}{\partial x} = \frac{\partial^2}{\partial x} f(x,y)$$

$$f_{yy}(x,y) = \frac{\partial^2 z}{\partial y} = \frac{\partial^2}{\partial y} f(x,y)$$

Notice how we can take the second partial derivative w.r.t y of $f_x(x,y)$ and vise versa. These work in the same way.

$$f_{xy}(x,y) = \frac{\partial}{\partial y} f_x(x,y)$$

$$f_{yx}(x,y) = \frac{\partial}{\partial x} f_y(x,y)$$

Notice how the subscripts work inside out.

33.0.1 Clairaut's Theorem

If we calculate $f_{xy}(x,y)$ and $f_{yx}(x,y)$, we'll notice they are equal to eachother.

$$\frac{\partial^2}{\partial x \partial y} f(x,y) = \frac{\partial^2}{\partial y \partial x} f(x,y)$$

or in similar notation $f_{xy}(x,y) = f_{yx}(x,y)$.

34 Tangent Planes

The tangent plane to a function $f(x,y)$ is the plane that passes through the point (a,b) and contains both x and y tangent lines (from the partial derivatives). The plane has the form

$$f(x,y) = m_1 x + m_2 y + d$$

where

$$m_1 = f_x(a,b) \quad \text{and} \quad m_2 = f_y(a,b)$$

34.0.1 Example

Find the tangent plane to $z = 4x^2 + y^2$ at the point $(1,2)$.

To start let's find m_1 and m_2

$$\begin{aligned} m_1 = f_x(1,2) &= \frac{\partial}{\partial x}(4x^2 + y^2) \\ &= 8x = 8(1) = 8 \end{aligned}$$

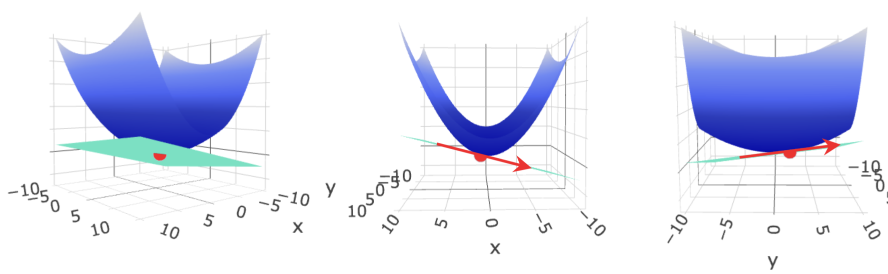
$$\begin{aligned} m_2 = f_y(1,2) &= \frac{\partial}{\partial y}(4x^2 + y^2) \\ &= 2y = 2(2) = 4 \end{aligned}$$

To find d, plug in the point $(1,\ 2,\ f(1,2))$.

$$\begin{aligned} z &= 8x + 4y + d \\ (8) &= 8(1) + 4(2) + d \\ d &= -8 \\ \therefore z &= 8x + 4y - 8 \end{aligned}$$

34.0.2 Visualization

This is what's really happening for the example we did. As you can see, the plane is tangent to the point in both the x and y perspective.



34.1 Linearization

We can equivalently express the tangent plane in the following form

$$L(x,y) = m_1(x - x_0) + m_2(y - y_0) + f(x_0, y_0)$$

Note, x_0, y_0 are our point coordinates and have replaced a, b. We primarily use linearization to estimate $\Delta f(x,y)$ near a point.

34.1.1 Example

Find the linearization of $f(x,y) = \sqrt{x^2 + 4y^2}$ at the point (3,2) and use it to approximate $f(2.9, 2.1)$.

First solve for m_1, m_2 and $f(x_0, y_0)$

$$m_1 = f_x(3,2) = \frac{3}{5}$$
$$m_2 = f_y(3,2) = \frac{8}{5}$$
$$f(x_0, y_0) = 5$$

Now we simply plug this into the formula to find the linearization.

$$L(x,y) = \frac{3}{5}(x-3) + \frac{8}{5}(y-2) + 5$$

$$L(2.9, 2.1) = \frac{3}{5}((2.9)-3) + \frac{8}{5}((2.1)-2) + 5$$
$$\therefore L(2.9, 2.1) = \frac{225}{50}$$

35 Differentials

We can *generalize* linearization to approximate any point

$$\Delta z \approx dz = f_x(x,y)dx + f_y(x,y)dy$$

This is called the (total) differential of the function or in other words, how the function changes to changes in x and y.

35.0.1 Example

The dimensions of a box are 5 ft, 7 ft, and 8 ft with an error in measurement of at most 0.1 ft. Estimate the maximum error in the calculated volume of the box.

We'll let $V(l, w, h) = l \cdot w \cdot h$ where $l = 5 \pm 0.1$, $w = 7 \pm 0.1$, $h = 8 \pm 0.1$.

$$\Delta V(l,w,h) = V_l(l,w,h)dl + V_w(l,w,h)dw + V_h(l,w,h)dh$$
$$\Delta V = m_1 dl + m_2 dw + m_3 dh$$

Now we can solve for m_1, m_2, and m_3.

$$m_1 = V_l(l,w,h) = 56$$
$$m_2 = V_w(l,w,h) = 40$$
$$m_3 = V_h(l,w,h) = 35$$

Since the maximum error is +0.1 ft for each we can substitute that here,

$$\Delta V = 56(0.1) + 40(0.1) + 35(0.1)$$
$$= \frac{131}{10}$$

36 Multivariable Chain Rule

To take the derivative of a multivariable function where the inputs are functions, we use this chain rule. If we have $f(x(t), y(t)) = z$ then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

but the easier way to remember this notation is understanding we take the partial derivative of each input as well as its own derivative so,

$$\frac{df(x,y)}{dt} = f_x(x,y)x' + f_y(x,y)y'$$

36.0.1 Example

Let $w(u,v) = f(x(u,v), y(u,v))$ with $x = 5\cos(u)+7\sin(v)$ and $y = 3\cos(u)\sin(v)$. If $f_x(0,0) = 8$ and $f_y(0,0) = 4$, find $w_u(\frac{\pi}{2}, 0)$.

Start by noting

$$\frac{dw}{du} = f_x(x,y)x' + f_y(x,y)y'$$

where $x(u,v)$ and $y(u,v)$ are being differentiated with respect to u. A more clear form of this is

$$\frac{dw}{du} = \frac{\partial(x(u,v), y(u,v))}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial(x(u,v), y(u,v))}{\partial y}\frac{\partial y}{\partial u}$$

Calculating x' and y' we find,

$$\begin{aligned} x' &= -5\sin(u) \\ y' &= -3\sin(v)\sin(u) \end{aligned}$$

evaluating these at $(\frac{\pi}{2}, 0)$ we get -5 and 0 respectively. Notice that $x(\frac{\pi}{2}, 0)$ and $y(\frac{\pi}{2}, 0)$ both equal 0. We can now substitute everything we know into our equation.

$$\begin{aligned} \left.\frac{dw}{du}\right|_{(u,v)=(\frac{\pi}{2},0)} &= f_x\left(x\left(\frac{\pi}{2},0\right), y\left(\frac{\pi}{2},0\right)\right)\cdot -5 + f_y\left(x\left(\frac{\pi}{2},0\right), y\left(\frac{\pi}{2},0\right)\right)\cdot 0 \\ &= f_x(0,0)\cdot -5 \\ &= 8\cdot 5 \\ &= -40 \end{aligned}$$

36.1 Implicit Differentiation

Another way to do implicit differentiation is by using the following formula

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

We first rewrite what we have as $F(x, y) = k$ (so bring all terms to one side), then we find the partial derivatives F_x and F_y, and lastly use the formula.

For functions with more than one independant variable like $F(x, y, z)$, then we use the following to implicitly differentiate

$$\frac{dz}{dx} = -\frac{F_x}{F_z}, \qquad \frac{dz}{dy} = -\frac{F_y}{F_z}$$

Notice how the partial just flips!

36.1.1 Example

Find dy/dx if $e^{xy^2} = x - 3y$.

First we rewrite as $F(x, y) = k$

$$F(x, y) = e^{xy^2} - x + 3y = 0$$

Then we get the partial derivatives

$$\begin{aligned} F_x(x, y) &= y^2 e^{xy^2} - 1 \\ F_y(x, y) &= 2xye^{xy^2} + 3 \end{aligned}$$

Finally we can use the formula!

$$\begin{aligned} \frac{dy}{dx} &= -\frac{y^2 e^{xy^2} - 1}{2xye^{xy^2} + 3} \\ &= \frac{1 - y^2 e^{xy^2}}{2xye^{xy^2} + 3} \end{aligned}$$

37 Gradient

The gradient is a vector field which points in the direction of **steepest** ascent. It is simply a *vector* that has the functions partial derivatives as its components.

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$$

We can use the functions gradient to find the **directional derivative** of a function at a point $\vec{x} = (x, y)$ in the direction of a unit vector $\vec{u}$.

$$D_{\vec{u}} f(x, y) = \nabla f(x, y) \cdot \vec{u}$$

Note that is dot product, not multiplication!

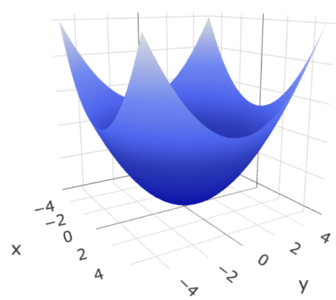
37.1 Magnitude

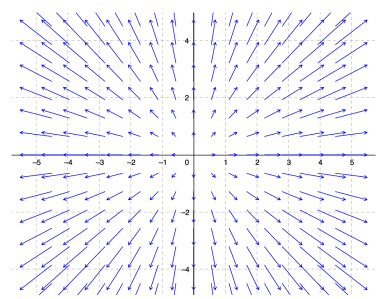
Recall how we can rewrite a dot product as

$$\begin{aligned} D_{\vec{u}} f(x,y) &= \nabla f(x,y) \cdot \vec{u} \\ &= \|\nabla f(x,y)\| \|\vec{u}\| \cos(\theta) \\ &= \|\nabla f(x,y)\| \cos(\theta) \end{aligned}$$

This tells us that $\|\nabla f(x,y)\|$ is the maximum rate of change when $\cos(0) = 1$ and $-\|\nabla f(x,y)\|$ when $\cos(\pi) = -1$ is the minimum rate of change and points us in the direction of steepest descent. This means the gradient of a function is always orthogonal to all counter lines and level sets!

37.1.1 Visualization





On the left we have the function $f(x,y) = x^2 + y^2$ and on the right we have its gradient which is the vector field $\langle 2x, 2y \rangle$. Notice how the vectors in the field all point in the direction of steepest ascent.

37.1.2 Key Reminders About Vectors

Recall the magnitude/norm of a vector $\vec{v}$ is given by

$$\|\vec{v}\| = \sqrt{{v_1}^2 + {v_2}^2 + \cdots + {v_n}^2}$$

and the unit vector is defined as a vector with magnitude 1 and can be normalized with the following

$$\hat{u} = \frac{\vec{u}}{\|\vec{u}\|}$$

where $\vec{u}$ is a non-zero vector.

37.1.3 Example

Find the directional derivative of $f(x, y) = \tan^{-1}(xy)$ at the point $(2, 3)$ in the direction parallel to the vector $\vec{v} = 5\vec{i} + 2\vec{j}$.

Start by noting the definition of a directional derivative

$$D_{\vec{u}} f(x, y) = \nabla f(x, y) \cdot \vec{u}$$

We'll start by finding the gradient ($\nabla f(x, y)$) first

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$$

Note that the partial derivatives are

$$f_x(x, y) = \frac{y}{1 + (xy)^2} \quad \text{and} \quad f_y(x, y)\frac{x}{1 + (xy)^2}$$

With this, we can plug in the point (2,3) into our gradient

$$\therefore \nabla f(2, 3) = \langle \frac{3}{37}, \frac{2}{37} \rangle$$

Now we can focus on our unit vector.

$$\begin{aligned}
\hat{u} &= \frac{\vec{v}}{\|\vec{v}\|} \text{ where } \vec{v} = 5\vec{i} + 2\vec{j} \\
&= \frac{5\vec{i} + 2\vec{j}}{\sqrt{25 + 4}} \\
&= \frac{5}{\sqrt{29}}\vec{i} + \frac{2}{\sqrt{29}}\vec{j}
\end{aligned}$$

Recall i, j are unit vectors and we can rewrite algebraic vectors with the following

$$\vec{v} = \langle a, b \rangle = a\vec{i} + b\vec{j}$$

So now we have

$$\begin{aligned}
D_{\vec{u}} f(2, 3) &= \langle \frac{3}{37}, \frac{2}{37} \rangle \cdot \langle \frac{5}{\sqrt{29}}, \frac{2}{\sqrt{29}} \rangle \\
&= \frac{19}{37\sqrt{29}}
\end{aligned}$$

38 Riemann Sums in $\mathbf{R}^3$

If we are given a **rectangular** region $[a, b] \times [c, d]$, we can use Riemann sums to approximate the area in R^3. Just like in R^2, we can take a sample point (left, middle, right) but now we can take it as upper left/right, lower left/right, and dead on middle. This is what acts as our (x_i, y_j) in our height. So we get

$$\text{Volume} \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i, y_j)\Delta x \Delta y$$

where m, n are our number of intervals along x, y respectively, (x_i, y_j) our sample point on our block, $\Delta x = \frac{b-a}{m}$, and $\Delta y = \frac{d-c}{n}$. Notice how $\Delta x \Delta y$ is the area of the base ΔA.

From this we can define the multivariable definite integral as such

$$\int_R f(x,y)dA = \iint_R f(x,y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i, y_j)\Delta x \Delta y$$

38.0.1 Example

Approximate the definite integral of $f(x, y) = 4x^2 - y^2$ on $[-1, 1] \times [-2, 2]$ using two sub-intervals in each direction and lower right sample points.

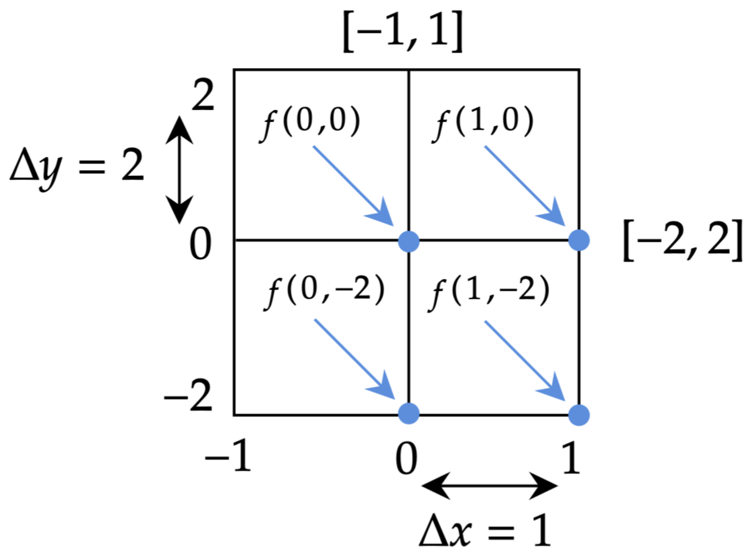
$$\int_{[-1,1]\times[-2,2]} f(x,y)\ dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i, y_j)\Delta x \Delta y$$

Our first step is to find Δx and Δy.

$$\begin{aligned} x \in [-1, 1], m &= 2 \text{ (\# of } x\text{-intervals)} \\ \Delta x &= \frac{b-a}{m} \\ &= \frac{1-(-1)}{2} = 1 \end{aligned}$$

$$\begin{aligned} y \in [-2, 2], n &= 2 \text{ (\# of } y\text{-intervals)} \\ \Delta y &= \frac{c-d}{n} \\ &= \frac{2-(-2)}{2} = 2 \end{aligned}$$

Since our problem is simple enough, we can create a diagram to analyze the lower right points.



$$\begin{aligned}\int_{[-1,1]\times[-2,2]} f(x,y)\ dA &\approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i,y_j)\Delta x\Delta y\\ &= \sum_{\text{blocks}} f(x_i,y_j)(2)\\ &= 2(f(0,0)+f(1,0)+f(0,-2)+f(1,-2))\\ &= 0\end{aligned}$$

Notice the total number of sample points is mn so in this case 4.

38.1 Fubini's Theorem

Since we have a **rectangular region**, the following is true

$$\int_R f(x,y)dA = \int_a^b\int_c^d f(x,y)\ dydx = \int_c^d\int_a^b f(x,y)\ dxdy$$

We evaluate these as partial integrals – we treat the other variables as constants just like in partial derivatives.

38.2 Separable Integrals

A special case of integral (similar to separable O.D.E's) is one of the form

$$\int_R f(x)g(y)\ d(A) = \left(\int_a^b f(x)\ dx\right)\left(\int_c^d g(y)\ dy\right)$$

We can do this as one of these functions acts completely as a constant, allowing us to bring it outside the working integral.

38.2.1 Example

Evaluate

$$\int_{-1}^{1}\int_{0}^{\pi/4} \frac{\sec^2(x)}{1+y^2}\ dxdy$$

We start by rewriting this as a separable integral

$$\begin{aligned}
&= \int_{-1}^{1}\int_{0}^{\pi/4} (\sec^2(x))\left(\frac{1}{1+y^2}\right)\ dxdy \\
&= \left(\int_{0}^{\pi/4} \sec^2(x)\ dx\right)\left(\int_{-1}^{1} \frac{1}{1+y^2}\ dy\right) \\
&= \left(\tan(x)\Big|_{0}^{\pi/4}\right)\left(\tan^{-1}(y)\Big|_{-1}^{1}\right) \\
&= \frac{\pi}{2}
\end{aligned}$$