2.2 Matrix Inverses

Definition: a $n \times n$ matrix A is **invertible** iff there exists $n \times n$ C such that:

$$AC = CA = I_n$$

where C is the inverse of A. Note that if A has an inverse, it is unique!

We denote the inverse of A as A^{-1}

We call a matrix **singular** if it is NOT invertible. If it is, we call it **nonsingular**

General Theorem for 2 $\times$ 2 Matrices:

$$\text{Let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \quad A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Generally Solving for Inverses: We want to know how to solve the equation $AB = I_n$ where B is the inverse of A. If A does not row reduce to I_n then it is not invertible!

$$\text{Row Reduce } \begin{bmatrix} A \mid I_n \end{bmatrix} \text{ until you get } \begin{bmatrix} I_n \mid A^{-1} \end{bmatrix}$$

Example: is $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ invertible?

$$\begin{bmatrix} A \mid I_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} I_3 \mid A^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{8} & \frac{-3}{8} & 1 \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{4} & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1/8 & -3/8 & 1 \\ 1/4 & 1/4 & -1 \end{bmatrix}$$

Properties: Assume A, A^{-1}, A^T and B are invertible

1. $(A^{-1})^{-1} = A$
2. $(AB)^{-1} = B^{-1}A^{-1}$
3. $(A^T)^{-1} = (A^{-1})^T$
4. $AA^{-1} = I_n$ and $A^{-1}A = I_n$

* * *

2.2 Elementary Matrices

a $n \times n$ matrix obtained by applying <u>one</u> rlementary row operation to I_n. If we multiply this to a matrix A, we are effectively doing the ERO on A.

Examples (2×2):

$$r \cdot R_2 \xrightarrow{+} R_1 = \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix}$$

$$\text{Swap } R_1 \text{ and } R_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\text{Rescale } R_1 \text{ by } a \in R = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$$

Examples (3×3):

$$r \cdot R_3 \xrightarrow{+} R_1 = \begin{bmatrix} 1 & 0 & r \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Swap } R_1 \text{ and } R_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Key Fact: $EA =$ a matrix obtained from A using the operation that defines E

By row reducing a matrix, we are left multiplying by a sequence of elementary matrices. In our discussions about inverses, if A row reduces to I_n then

$$(E_r \cdot E_{r-1} \ldots E_2 \cdot E_1) \cdot A = I_n$$

Using our previous definition of $A^{-1}A = I_n$, we can conclude that $A^{-1} = E_r \cdot E_{r-1} \ldots E_2 \cdot E_1$.

Example: $A = \begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}$. Find a formula for its inverse using elementary matrices.

$$R_1 \xrightarrow{swap} R_2 \quad \begin{bmatrix} 1 & 5 \\ 3 & -2 \end{bmatrix} \quad \left(E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right)$$

$$3 \cdot R_1 \xrightarrow{-} R_2 \quad \begin{bmatrix} 1 & 5 \\ 0 & -17 \end{bmatrix} \quad \left(E_2 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \right)$$

$$-\frac{1}{17} \cdot R_2 \quad \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \quad \left(E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{17} \end{bmatrix} \right)$$

$$5 \cdot R_2 \xrightarrow{-} R_1 \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \left(E_4 = \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix} \right)$$

$$\therefore A^{-1} = E_4 \cdot E_3 \cdot E_2 \cdot E_1 \text{ in this order}$$

* * *

2.3 Characterizations of Invertible Matrices

Let's talk about everything we can conclude about a matrix if it is invertible. Note, if one of these properties hold true about any matrix, it means it must be invertible as well.

When we talk about these properties, assume A is $m \times n$ and invertible:

1. A is row equivalent to I_n ($RREF(A) = I_n$)
2. From 1 we know RREF of A has n leading one's.
3. $Ax = 0$ has the only trivial solution ($x = 0$).
4. From 3, the columns of A are linearly independent.
5. From 4, the linear transformation $T(x) = Ax$ is one-to-one and onto
6. $Ax = b$ has atleast one solution for b
7. Columns of A span R^n

2.4 Partitioned Matrices

Consider matrices that split up into natural looking blocks. We can rewrite them to consist of smaller blocks. Assuming these blocks have appropiate sizes, we can do multiplication in block form.

Example:

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \text{with each } A_{ij} \text{ being } 3 \times 3$$

In this example, A^2 would look like:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} (A_{11})^2 + A_{12}A_{21} & A_{11}A_{12} + A_{12}A_{22} \\ A_{21}A_{11} + A_{22}A_{21} & A_{21}A_{12} + (A_{22})^2 \end{bmatrix}$$

More generally, this works for any type of block decomposition assuming all matrix products are defined!

Inverses: If a partioned matrix is in upper-triangular form, we can find its inverse which will also happen to be in upper-triangular form.

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \text{ therefore } A^{-1} = B = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix}$$

Since $AA^{-1} = I_n$, we can conclude $AB = I_{nm}$

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} & A_{11}B_{12} + A_{12}B_{22} \\ 0 & A_{22}B_{22} \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_m \end{bmatrix}$$

This means we want

1. $A_{11}B_{11} = I_n$
2. $A_{11}B_{12} + A_{12}B_{22} = 0$
3. $A_{22}B_{22} = I_m$

Equations 1 and 2 imply $B_{11} = (A_{11})^{-1}$ and $B_{22} = (A_{22})^{-1}$.
Using algebra, equation 3 will result in $B_{12} = -(A_{11})^{-1}A_{12}(A_{22})^{-1}$

$$\therefore\ A^{-1} = \begin{bmatrix} (A_{11})^{-1} & -(A_{11})^{-1}A_{12}(A_{22})^{-1} \\ 0 & (A_{22})^{-1} \end{bmatrix}$$

Example: Compute the inverse of A.

$$A = \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & -5 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix}$$

$$\text{Let's break this into blocks } A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$A_{11} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},\ A_{12} = \begin{bmatrix} 4 & 5 \\ -5 & 4 \end{bmatrix},\ A_{22} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$\text{Using our definition of } 2 \times 2 \text{ inverses, } (A_{11})^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix},\ (A_{22})^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$-(A_{11})^{-1}A_{12}(A_{22})^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 5 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & -3 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -3 \\ 3 & 4 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} (A_{11})^{-1} & -(A_{11})^{-1}A_{12}(A_{22})^{-1} \\ 0 & (A_{22})^{-1} \end{bmatrix}$$

$$\therefore\ A^{-1} = \begin{bmatrix} 1 & -2 & 8 & -3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

2.7 Applications to Computer Graphics

Computer graphics are stored as lists of vectors that contain the x and y coordinate of the vertex. We can manipulate this data and apply transformations by using matrix multiplication.

In order to encode movement, we need to do transformations without fixed points, but we get around this using **homogenous** coordinates.

$$\text{Replace } v \in R^n \text{ by } \begin{bmatrix} v \\ 1 \end{bmatrix} \in R$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \in R^2 \to \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \in R^3 \to \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in R^3 \to \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \in R^4$$

If we want to shift $\begin{bmatrix} x \\ y \end{bmatrix}$ by $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ using a homogenous transformation, we can do the following:

$$\begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+\alpha \\ y+\beta \\ 1 \end{bmatrix}$$

Similarly, we can do this in R^3 as well:

$$\begin{bmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 1 & \gamma \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x+\alpha \\ y+\beta \\ z+\gamma \\ 1 \end{bmatrix}$$

Composite Transformations: We can do two transformations in one (e.g, rotating and shifting). We can accomplish this by completing two matrix multiplications.

Let's say we want to rotate then shift, we can use the following matrix multiplication:

$$= \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: The right matrix will go first, so using this it will rotate then shift.

$$= \begin{bmatrix} \cos\theta & -\sin\theta & \alpha \\ \sin\theta & \cos\theta & \beta \\ 0 & 0 & 1 \end{bmatrix}$$

3.1 Determinants

If you recall the definition of the inverse of 2×2 matrices, you would've remembered the $\frac{1}{ad-bc}$ portion of it. The $ad - bc$ is called the determinant of the matrix and can be denoted by $det(A)$ or $|A|$.

The factor of which a linear transformation changes any area is called the determinant.

How to solve: We can solve for the determinant recursively using this formula:

$$\sum_{j=i}^{n}(-1)^{i+j} \cdot a_{ij} \cdot |A_{ij}|$$

Let's define some things:

- A_{ij} is A with row i and column j **removed**
- a_{ij} is the term at the i^{th} row, j^{th} column
- The (i, j) **cofactor** of a martix A to be $C_{ij} = (-1)^{i+j} \cdot |A_{ij}|$

We can cofactor expand along any row or column. How do we choose which one? Typically, the column or row with most 0's is favourable as $a_{ij} = 0$ which simplifies our computation.

Note: a matrix with a zero column or row has determinant = 0.

Similarly to partioned matrices, matrices in upper or lower triangular form are easy to compute the determinant for. We simply take the **product of the diagonal entries.**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{bmatrix} \text{ then } |A| = 1 \cdot 5 \cdot 9 = 45$$

3.2 Determinant Properties

Computing the determinant for upper-triangular or lower-triangular matrices is much easier than cofactor expansion. So why not row reduce to that point? Let's see how our typical ERO impact the determinant:

Type 1: add a multiple of one row to a different row:

- $|A| = |B|$

Type 2: swapping rows:

- $|A| = -|B|$

Type 3: rescaling by a non-zero scalar (α) then:

- $|B| = \alpha|A|$

The most important property about determinants:

$$|A| \neq 0 \iff A \text{ is invertible}$$

Other Core Properties:

1. $|A| \neq 0$ iff its rows are linearly independent
2. $|A| \neq 0$ iff its columns are linearly independent
3. $|A^T| = |A|$
4. $|AB| = |A| \cdot |B|$
5. $|A^n| = |A|^n$
6. $|A||A^{-1}| = 1$ so $|A^{-1}| = \frac{1}{|A|}$ and vise versa
7. $|\alpha A| = \alpha^n |A|$ where $\alpha \in R$ and n is the size of A
8. $adj(A) = |A|^{n-1}$
9. $\begin{vmatrix} a_1 + x_1 & b_1 & c_1 \\ a_2 + x_2 & b_2 & c_2 \\ a_3 + x_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x_1 & b_1 & c_1 \\ x_2 & b_2 & c_2 \\ x_3 & b_3 & c_3 \end{vmatrix}$

3.3 Cramer's Rule

We can solve $Ax = b$ using determinants. We can find $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ with $x_j = \dfrac{|A_j(b)|}{|A|}$.

How do we get $A_j(b)$? Replacing the j^{th} column with b. So for a 2×2 matrix, the general solution would be:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \dfrac{|A_1(b)|}{|A|} \\ \dfrac{|A_2(b)|}{|A|} \end{bmatrix}$$

Using a few key properties we can actually determine that the (i, j) entry of A^{-1} is $\dfrac{|A_i(e_j)|}{|A|}$.

Again, using the same system of cofactors, we can generalize finding A^{-1}. This leads us to the general inverse formula:

$$C_{ji} = (-1)^{i+j} \cdot |A_i(e_j)|$$

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} c_{11} & c_{21} & \dots & c_{n1} \\ c_{12} & c_{22} & \dots & c_{n2} \\ \vdots & \vdots & & \vdots \\ c_{1n} & c_{2n} & \dots & c_{nn} \end{bmatrix}$$

An easier way to calculate the c_{ji} terms is to calculate $\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$ using our typical definition of cofactors, and then transpose it after.

Example: Find c_{11} from A.

$$A = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 4 & 1 \\ 9 & 1 & 0 \end{bmatrix}$$

$$c_{11} = (-1)^{1+1} \cdot \begin{vmatrix} 4 & 1 \\ 1 & 0 \end{vmatrix}$$

Once we've calculated all the c_{ij} terms, we can then transpose:

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}^T$$

4.1 Vector Spaces

A vector space is a set of elements we call vectors. They must abide by vector addition and scalar multiplication. These operations satisfy the 10 axioms. Vector spaces typically look like vector coordinates, martices, and polynomials.

Subspace: a subspace of a vector space must follow these 3 axioms:

- $\vec{0}$ ∈H → the 0 vector is in the subset
- The subset is closed under vector addition
- The subset is closed under scalar multiplication

4.2 Nullspace, Column Space, Row Space

Nul(A): The nullspace of A is the **set** of vectors such that $Ax = 0$. We find its solutions through the parametric solution to a homogenous system.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

We see that x_3 is a free parameter $\text{Nul}(A) = \{x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} | x_3 \in R\} = \text{span}\{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\}$

Col(A): The column space of A is the span of columns of A. To test if $b \in \text{Col}(A)$, augment b to A, row reduce, and if the system is consistent, $b \in$ Col(A)

Row reducing A alone will tell us if Col(A)=R^m. If each row has a leading 1, then Col(A)=R^m.

$$\text{Is } b = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \text{ in Col} \begin{bmatrix} 3 & 2 \\ 4 & 2 \\ -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 & 2 \\ 4 & 2 & 2 \\ -1 & 0 & -1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\therefore \text{ this system is inconsistent, } b \notin \text{Col}(A)$$

Row(A): The row space of A is the span of the rows of A (similar to Col(A)). Notice that:

$$\text{Row}(A) = \text{Col}(A^T)$$

Transformations: Taking a look at linear transformations $T(x) = Ax$, we can apply what we just learned to them.

$$\text{Ker(T)} = \{x \in V | T(x) = 0\} = \text{Nul}(A)$$
$$\text{Range(T)} = T(y) = \text{Col(A)}$$

4.3 Bases

A basis for H (a subset of a vector space) holds for the following:

- The basis is linearly independent
- Span(B) = H

Here's how we compute the basis for Nul(A), Col(A), and Row(A).

Nul(A): row reduce and write the general solution for $Ax = 0$. Those parametric solutions are your basis vectors for Nul(A).

Col(A): row reduce and consider the columns with leading 1's in the RREF. Those **corresponding** columns in A are a basis for Col(A).

Row(A): row reduce and the non-zero rows are a basis for Row(A).

Example: Find basis for Nul(A), Col(A), and Row(A).

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

$$RREF(A) = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$Nul(A) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore \{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \} \text{ is a basis for Nul}(A)$$

Columns 1 and 2 have leading ones in them, so

$$\therefore \{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \} \text{ are a basis for Col}(A)$$

Row 1 and 2 are non-zero, so

$$\therefore \{ \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix} \} \text{ are a basis for Row}(A)$$

4.4 Coordinate System

Every vector in a vector space can be written as a linear combination:

$$v = c_1 v_1 + c_2 v_2 + ... + c_n v_n$$

We can get the coordinate vector of v by using this:

$$[v]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \in R$$

Example:

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \text{ is invertible, so } B = \{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\} \text{ is a basis for } R^2\text{.If } v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \text{ what is } [v]_B$$

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \iff \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$c_1 = 1 \text{ and } c_2 = -1$$

$$\therefore \text{ for this basis } B, \, [v]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Polynomials: We can write their basis as $B = \{1, x, x^2, x^3, ..., x^n\}$. For example if $v = 1 + x - 5x^2 + 2x^3 \longrightarrow [v]_B = \begin{bmatrix} 1 \\ 1 \\ -5 \\ 2 \end{bmatrix}$

Changing Coordinates: Solving the system

$$\begin{bmatrix} v_1 & v_2 & ... & v_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = x$$

We can see that the solution $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ is $[x]_b$.

If we call $P_B = \begin{bmatrix} v_1 & v_2 & ... & v_n \end{bmatrix}$, then we can derive this formula:

$$P_B \cdot [x]_B = x \longrightarrow [x]_B = P_B^{-1} \cdot x$$

Using this, we can easily compute new coordinate vectors.

Example: We are using a random P_B's inverse.

$$y = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$$

$$[y]_B = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}$$

6.1 Dot Product

$$u \cdot v = u_1 v_1 + u_2 v_2 + ... + u_n v_n$$

$$\begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ -2 \\ -1 \\ -4 \end{bmatrix} = (4)(-3) + (3)(-2) + (2)(-1) + (1)(-4) = -24$$

Length/norm of v: $||v|| = \sqrt{v \cdot v}$. If we have $||cV|| = |c|||V||$.

Vectors with length 1 are called unit vectors. It is only a unit vector if its on the unit circle.

For example, $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ is a unit vector in R^2. In R^3, $\begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ are unit vectors.

Example: Find a constant c so that cV is a unit vector:

$$V = \begin{bmatrix} 1 \\ -3 \\ 2 \\ 1 \end{bmatrix}$$

Let's compute $||cV||^2$ and choose c so it equals 1

$$||cV||^2 = (cV) \cdot (cV) = c^2(V \cdot V)$$

$$1 = c^2(1 + 9 + 4 + 1)$$

$$1 = 15c^2$$

$$c = \pm\sqrt{\frac{1}{15}}$$

Example: What is the distance between $u = \begin{bmatrix} 10 \\ 0 \\ 3 \\ -1 \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ 2 \\ -4 \\ 3 \end{bmatrix}$

$$d(u, v) = || \begin{bmatrix} 10 \\ 0 \\ 3 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ -4 \\ 3 \end{bmatrix} ||$$

$$= || \begin{bmatrix} 9 \\ -2 \\ 7 \\ -4 \end{bmatrix} ||$$

$$= \sqrt{18 + 4 + 49 + 16}$$

$$= \sqrt{150}$$

Orthogonal: if $u \cdot v = 0$. u and v are orthogonal <u>iff</u> $||u + v||^2 = ||u||^2 + ||v||^2$

$W \in R^n$ is a subspace, then the orthogonal complement is $W^{\perp} = \{v \in R^n | v \cdot w = 0\}$.

Example: Show that a vector $\in W^{\perp}$

$$W = span\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \}$$

$$V = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Check the following dot products are 0. Dot v with each vector in the span

$$v \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = (1)(1) + (-2)(2) + (1)(3) = 0$$

$$v \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = (1)(3) + (-2)(2) + (1)(1) = 0$$

$$\therefore v \in W^{\perp}$$

Angles between u and v: $u \cdot v = ||u||||v|| \cos\theta$ so $\theta = \cos^{-1}\left(\frac{u \cdot v}{||u||||v||}\right)$

Orthogonal Sets: if $v_i \cdot v_j = 0$ for all $i \neq j$. For example $\{ e_1, e_2, ..., e_n \}$ is an orthogonal set.